

AN ALTERNATIVE RECURSIVE FORMULA FOR THE SUMS OF POWERS OF INTEGERS

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ABSTRACT. In this note, we derive an alternative recursive formula for the sums of powers of integers involving the Stirling numbers of the first kind. As a remarkable by-product, we provide a non-recursive definition of the Catalan numbers.

1. INTRODUCTION

The sums of powers of the first n positive integers $S_p(n) = 1^p + 2^p + \cdots + n^p$, ($p = 0, 1, 2, \dots$) satisfy the fundamental identity

$$1 + \sum_{t=0}^p \binom{p+1}{t} S_t(n) = (n+1)^{p+1}, \quad p \geq 0, \quad (1)$$

from which we can successively compute $S_0(n)$, $S_1(n)$, $S_2(n)$, etc. Identity (1) can easily be proved by using the binomial theorem; see, e.g., [1, 2]. Several variations of (1) are also well known [3, 4, 5].

In this note we derive the following lesser-known recursive formula for $S_p(n)$

$$p! + \sum_{t=0}^p \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right] S_t(n) = p! \binom{n+p+1}{p+1}, \quad p \geq 0, \quad (2)$$

where $\left[\begin{matrix} p \\ t \end{matrix} \right]$ denote the (unsigned) Stirling numbers of the first kind, also known as the Stirling cycle numbers (see, e.g., [6, Chapter 6]). Although the recursive formula (2) is by no means new (a slightly different form of it can be obtained, for example, from the results of Ref. [7]), our purpose in dealing with recurrence (2) in this note is two-fold. On one hand, we aim to provide a new algebraic proof of (2) by making use of two related identities involving the harmonic numbers. On the other hand, as we will show, using (2) in conjunction with the principle of strong mathematical induction yields the neat identity

$$\sum_{t=0}^p r^t \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right] = \frac{(p+r)!}{r!}, \quad (3)$$

which holds for every $r = 1, 2, 3, \dots$. Interestingly enough, for the special case in which $r = p$, identity (3) leads right away to an explicit formula for the Catalan numbers that seems not to have been noticed hitherto (see equation (10) below).

Key words and phrases. Sums of powers of integers, Stirling numbers of the first kind, Catalan numbers.

2. DERIVATION OF THE RECURSIVE FORMULA

To prove formula (2), we proceed in two steps. In the first step, we state the identity:

• **IDENTITY 1**

$$\sum_{k=1}^n \binom{k+p}{p} H_k = \binom{n+p+1}{p+1} H_n - \frac{1}{(p+1)!} \sum_{t=0}^p \begin{bmatrix} p+1 \\ t+1 \end{bmatrix} S_t(n),$$

where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the k -th harmonic number. In the second step, we state the identity:

• **IDENTITY 2**

$$\sum_{k=1}^n \binom{k+p}{p} H_k = \binom{n+p+1}{p+1} H_n - \frac{1}{p+1} \left[\binom{n+p+1}{p+1} - 1 \right].$$

Formula (2) then follows by equating the right-hand sides of Identities 1 and 2.

2.1. Proof of the Identity 1. To prove Identity 1, we shall use the following lemma.

Lemma 1. *For any non-negative integer p and for $t = 0, 1, \dots, p+1$, we have*

$$\sum_{k=t}^{p+1} (-1)^{p+1-k} p^{k-t} \binom{k}{t} \begin{bmatrix} p+1 \\ k \end{bmatrix} = \begin{bmatrix} p+1 \\ t \end{bmatrix}. \quad (4)$$

Proof. Let $[x]_p$ denote the falling factorial $x(x-1)(x-2)\cdots(x-p+1)$. Recall that the numbers $\begin{bmatrix} p \\ t \end{bmatrix}$ can be defined algebraically by the relation [6, Equation (6.11)]

$$[x]_p = \sum_{k=0}^p (-1)^{p-k} \begin{bmatrix} p \\ k \end{bmatrix} x^k.$$

Then we can evaluate $[x+p]_{p+1}$ as

$$\begin{aligned} [x+p]_{p+1} &= \sum_{k=0}^{p+1} (-1)^{p+1-k} \begin{bmatrix} p+1 \\ k \end{bmatrix} (x+p)^k \\ &= \sum_{k=0}^{p+1} (-1)^{p+1-k} \begin{bmatrix} p+1 \\ k \end{bmatrix} \sum_{t=0}^k \binom{k}{t} p^{k-t} x^t \\ &= \sum_{t=0}^{p+1} \sum_{k=t}^{p+1} (-1)^{p+1-k} p^{k-t} \binom{k}{t} \begin{bmatrix} p+1 \\ k \end{bmatrix} x^t. \end{aligned}$$

On the other hand, $[x+p]_{p+1}$ can alternatively be written as

$$[x+p]_{p+1} = [x]^{p+1} = \sum_{t=0}^{p+1} \begin{bmatrix} p+1 \\ t \end{bmatrix} x^t,$$

where $[x]^p$ denotes the rising factorial $x(x+1)(x+2)\cdots(x+p-1)$. Therefore, equating coefficients of x^t on the right-hand sides of the last two equations, we end up with relation (4). \square

Next we proceed with the proof of the Identity 1:

$$\begin{aligned}
\sum_{k=1}^n \binom{k+p}{p} H_k &= \sum_{k=1}^n \sum_{j=1}^k \binom{k+p}{p} j^{-1} = \sum_{j=1}^n \sum_{k=j}^n \binom{k+p}{p} j^{-1} \\
&= \sum_{j=1}^n j^{-1} \sum_{k=1}^n \binom{k+p}{p} - \sum_{j=1}^n j^{-1} \sum_{k=1}^{j-1} \binom{k+p}{p} \\
&= \binom{k+p+1}{p+1} H_n - \sum_{j=1}^n \binom{j+p}{p+1} j^{-1}. \tag{5}
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{j=1}^n \binom{j+p}{p+1} j^{-1} &= \sum_{j=1}^n j^{-1} \frac{1}{(p+1)!} \sum_{k=0}^{p+1} (-1)^{p+1-k} \begin{bmatrix} p+1 \\ k \end{bmatrix} (j+p)^k \\
&= \frac{1}{(p+1)!} \sum_{t=0}^{p+1} \sum_{k=t}^{p+1} (-1)^{p+1-k} p^{k-t} \binom{k}{t} \begin{bmatrix} p+1 \\ k \end{bmatrix} \sum_{j=1}^n j^{t-1}.
\end{aligned}$$

By relation (4), this reduces to

$$\sum_{j=1}^n \binom{j+p}{p+1} j^{-1} = \frac{1}{(p+1)!} \sum_{t=0}^{p+1} \begin{bmatrix} p+1 \\ t \end{bmatrix} S_{t-1}(n).$$

Since $\begin{bmatrix} p+1 \\ 0 \end{bmatrix} = 0$, this is in turn equivalent to

$$\sum_{j=1}^n \binom{j+p}{p+1} j^{-1} = \frac{1}{(p+1)!} \sum_{t=0}^p \begin{bmatrix} p+1 \\ t+1 \end{bmatrix} S_t(n). \tag{6}$$

Finally, combining equations (5) and (6) gives Identity 1.

2.2. Proof of the Identity 2. To prove Identity 2, we employ the following version of Abel's lemma on summation by parts (see e.g. [8]).

Lemma 2 (Abel's lemma). *Let $\{u_k\}_{k \geq 1}$ and $\{v_k\}_{k \geq 1}$ be two sequences of real numbers with partial sums $U_n = \sum_{k=1}^n u_k$ and $V_n = \sum_{k=1}^n v_k$. Further define $U_0 = V_0 = 0$. Then the following relation holds true:*

$$\sum_{k=1}^n u_k V_k + \sum_{k=1}^n v_k U_{k-1} = U_n V_n. \tag{7}$$

Hence, letting

$$v_k = \frac{1}{k}, \quad V_k = \sum_{j=1}^k \frac{1}{j} = H_k,$$

$$u_k = \binom{k+p}{p}, \quad U_k = \sum_{j=1}^k \binom{j+p}{p} = \binom{k+p+1}{p+1} - 1,$$

and plugging into (7) yields

$$\sum_{k=1}^n \binom{k+p}{p} H_k = \binom{n+p+1}{p+1} H_n - \sum_{k=1}^n \frac{1}{k} \binom{k+p}{p+1}.$$

Futhermore, it is clear that

$$\sum_{k=1}^n \frac{1}{k} \binom{k+p}{p+1} = \frac{1}{p+1} \sum_{k=1}^n \binom{k+p}{p} = \frac{1}{p+1} \left[\binom{n+p+1}{p+1} - 1 \right],$$

and thus, from the last two equations, Identity 2 follows.

3. CATALAN NUMBERS ENTER THE SCENE

First we note that formula (2) can be written in the equivalent form

$$\sum_{t=0}^p r^t \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right] = p! \left[\binom{r+p+1}{p+1} - 1 \right] - \sum_{j=1}^{r-1} \sum_{t=0}^p j^t \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right], \quad (8)$$

where r stands for any arbitrary fixed positive integer. In particular, for $r = 1$ we retrieve the well-known relation for the Stirling cycle numbers

$$\sum_{t=0}^p \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right] = (p+1)!,$$

which constitutes the base case of the above identity (3). Let us assume as a strong inductive hypothesis that

$$\sum_{t=0}^p j^t \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right] = \frac{(p+j)!}{j!}, \quad \text{for } j = 1, 2, \dots, r-1. \quad (9)$$

Thus, substituting (9) into (8), we have

$$\begin{aligned} \sum_{t=0}^p r^t \left[\begin{matrix} p+1 \\ t+1 \end{matrix} \right] &= p! \left[\binom{r+p+1}{p+1} - 1 \right] - \sum_{j=1}^{r-1} \frac{(p+j)!}{j!} \\ &= p! \left[\binom{r+p+1}{p+1} - 1 \right] - p! \sum_{j=1}^{r-1} \binom{p+j}{j} \\ &= p! \left[\binom{r+p+1}{p+1} - \binom{r+p}{p+1} \right] = \frac{(p+r)!}{r!}. \end{aligned}$$

This completes the inductive step and the proof of the identity (3).

Observe that, for $r = p$, the said identity becomes

$$\sum_{t=0}^p p^t \begin{bmatrix} p+1 \\ t+1 \end{bmatrix} = (p+1)!C_p,$$

where $C_p = \frac{1}{p+1} \binom{2p}{p}$ is the p -th Catalan number [9]. Expressing C_p as

$$C_p = \frac{\sum_{t=1}^{p+1} p^{t-1} \begin{bmatrix} p+1 \\ t \end{bmatrix}}{\sum_{t=1}^{p+1} \begin{bmatrix} p+1 \\ t \end{bmatrix}}, \quad p \geq 1, \quad (10)$$

we can therefore interpret C_p as the average of the function p^{t-1} over all $(p+1)!$ permutations of $p+1$ elements, with t being the number of cycles of a permutation, and $\begin{bmatrix} p+1 \\ t \end{bmatrix}$ the number of permutations of $p+1$ elements with exactly t cycles. As an illustrative example, we apply (10) to get C_6 :

$$\begin{aligned} C_6 &= \frac{1}{7!} \left(6^0 \begin{bmatrix} 7 \\ 1 \end{bmatrix} + 6^1 \begin{bmatrix} 7 \\ 2 \end{bmatrix} + 6^2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + 6^3 \begin{bmatrix} 7 \\ 4 \end{bmatrix} + 6^4 \begin{bmatrix} 7 \\ 5 \end{bmatrix} + 6^5 \begin{bmatrix} 7 \\ 6 \end{bmatrix} + 6^6 \begin{bmatrix} 7 \\ 7 \end{bmatrix} \right) \\ &= \frac{1}{5040} (6^0 \cdot 720 + 6^1 \cdot 1764 + 6^2 \cdot 1624 \\ &\quad + 6^3 \cdot 735 + 6^4 \cdot 175 + 6^5 \cdot 21 + 6^6 \cdot 1) = \frac{665280}{5040} = 132. \end{aligned}$$

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